# ROBUST FILTERING BASED ON PROBABILISTIC DESCRIPTIONS OF MODEL ERRORS

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**Abstract.** A new approach to robust estimation of signals and prediction of time–series is considered. Signal and system parameter deviations are represented as random variables, with known covariances. A robust design is obtained by minimizing the squared estimation error, averaged both with respect to model errors and noise. A polynomial solution, based on averaged spectral factorizations and averaged Diophantine equations, is derived. The robust estimator is called a cautious Wiener filter. It turns out to be no more complicated to design than an ordinary Wiener filter.

**Keywords:** Robustness; Optimal estimation; Wiener filtering; Deconvolution; Prediction.

## 1. INTRODUCTION

Kalman and Wiener filters are in frequent use. With many computer programs for design available, filter parameters are easy to obtain. Perhaps too easy, since we are not forced to specify the validity of our models. It is also hard to take a whole range of expected system behaviour into account. The goal of *robust filter* synthesis is to overcome these drawbacks.

We here propose a novel approach to robust design for signal estimation. It is based on a stochastic description of model errors, related to the stochastic embedding concept of Goodwin and Salgado (1989). A single robust filter, for the whole class of possible models, is obtained by minimizing the squared estimation error, averaged both with respect to model errors and the noise.

Most previous suggestions for robust filter design have been based on the minimax approach. See e.g. D'Appolito and Hutchinson (1972) and Kassam and Poor (1985). Apart from leading to a much simpler design methodology, the approach proposed here avoids two drawbacks of robust minimax design. First, the descriptions of model uncertainties may have *soft bounds.* These are more readily obtainable in a noisy environment than the hard bounds required for minimax design. Secondly, not only the range of uncertainties, but also their likelihood is taken into account; Probable model errors will have a greater impact on an estimator design than do very rare "worst cases". The conservativeness is thus reduced.

A polynomial solution, based on averaged spectral factorizations and averaged Diophantine equations, will be presented. The design procedure constitutes a generalization of the polynomial equations approach, which was pioneered by Kučera (1979). Mild solvability conditions guarantee the existence of stable optimal filters. The robust design turns out to be no more complicated than the design of an ordinary Wiener filter. The methodology is here exemplified on a scalar discrete–time deconvolution problem. Derivation of design equations for this and other problems, such as state estimation and feedforward control, can be found in Sternad and Ahlén (1993).

**Remarks on the notation.** For any complex polynomial of degree $np$, in the backward shift operator

$$P(q^{-1}) = p_o + p_1 q^{-1} + \ldots + p_{np} q^{-np}$$

the *conjugate polynomial* is defined as $P_*(q) \triangleq p_o^* + p_1^* q + \ldots + p_{np}^* q^{np}$. For convenience, polynomial arguments will often be omitted. We call $P(q^{-1})$ *stable* if all zeros of $P(z^{-1})$ are in $|z| < 1$.

## 2. THE ESTIMATION PROBLEM

A scalar deconvolution problem will be considered, to illustrate the design principles. It includes eg. ordinary output filtering and prediction of ARMA–processes as special cases. It also includes the design of linear recursive equalizers for digital communications. Measurements are described as

$$y(t) = \mathcal{G}(q^{-1}) u(t-k) + w(t) \quad . \tag{1}$$

The linear, causal and possibly uncertain transfer function $\mathcal{G}(q^{-1})$ may eg. represent a transducer. If the delay is uncertain, $k$ denotes its minimum value. The input $u(t)$ and the measurement noise $w(t)$ are described by possibly uncertain ARMA–models

$$u(t) = \mathcal{F}(q^{-1}) e(t) \quad ; \quad w(t) = \mathcal{H}(q^{-1}) v(t) \tag{2}$$

$$E|e(t)|^2 = 1 \quad ; \quad E|v(t)|^2 \triangleq \sigma_v^2 \quad .$$

The time–series $e(t)$ and $v(t)$ are assumed mutually uncorrelated. They are stationary white noises or impulse sequences, with zero mean. The variance $\sigma_v^2$ may be uncertain and has nominal value $\rho_0$. All transfer functions are assumed time–invariant. Signals may be complex–valued; this is the case eg. in digital communications applications.

A stable, linear and time–invariant estimator of $u(t)$, given $y(t+m)$, is sought. Depending on $m$, it could be a predictor ($m<0$), a filter ($m=0$) or a fixed lag smoother ($m>0$).

## 3. ADDITIVE PROBABILISTIC ERROR MODELS

The transfer functions $\mathcal{G}$, $\mathcal{F}$ and $\mathcal{H}$ may be uncertain. An *error model* is a quantification of the model error class. Together with a nominal model, it constitutes the *extended design model* on which a robust design is based. As error models, we will utilize additive transfer functions $\Delta\mathcal{F}, \Delta\mathcal{G}, \Delta\mathcal{H}$, with unknown stochastic numerators and pre–specified denominators. This choice is crucial for obtaining a simple solution to the filtering problem. The extended design models are specified as

$$\begin{aligned}
\mathcal{F} &= \frac{C_0}{D_0}+\frac{C_1\Delta C}{D_1} = \frac{C_0D_1+D_0C_1\Delta C}{D_0D_1} \triangleq \frac{C}{D} \\
\mathcal{G} &= \frac{B_0}{A_0}+\frac{B_1\Delta B}{A_1} = \frac{B_0A_1+A_0B_1\Delta B}{A_0A_1} \triangleq \frac{B}{A} \\
\mathcal{H} &= \frac{M_0}{N_0}+\frac{M_1\Delta M}{N_1} = \frac{M_0N_1+N_0M_1\Delta M}{N_0N_1} \triangleq \frac{M}{N}
\end{aligned} \tag{3}$$

Above, $C_0/D_0$ etc. represent nominal models, with degrees $nc_0, nd_0$ etc. They are assumed known and stable. Stable "error denominators" $D_1, A_1$ and $N_1$, of degrees $nd_1, na_1$ and $nn_1$, as well as numerator factors $C_1, B_1, M_1$, of degrees $nc_1, nb_1$ and $nm_1$, may be specified by the designer or obtained from data. Coefficients of numerator polynomials

$$\Delta P(q^{-1}) = \Delta p_0 + \Delta p_1 q^{-1} + \ldots + \Delta p_{\delta p} q^{-\delta p} \tag{4}$$

are stochastic variables. They have zero means and parameter covariances $\bar{E}\Delta p_i \Delta p_j^*$, collected in the covariance matrices $\mathbf{P}_{\Delta P}$. These coefficients are constant in time, so they are independent of the time series $e(t)$ and $v(t)$. Except for first and second order moments, their distributions need not be known, since they will not affect the design.

All polynomial degrees are assumed known to (or specified by) the designer [1]. Note also that denominators polynomials are *all assumed stable.* Model uncertainty more or less forces us to restrict attention to stable extended design models [2].

In the following, we utilize three further assumptions:

**A1.** Coefficients of $\Delta C$ and of $\Delta B$ are independent.

**A2.** The covariance matrices $\mathbf{P}_{\Delta C}, \mathbf{P}_{\Delta B}$ and $\mathbf{P}_{\Delta M}$ are Hermitian and positive semidefinite.

**A3.** The standard deviation of $v(t), \sigma_v$, is regarded as a stochastic variable, independent of $\Delta M$, with mean $\sqrt{\rho_0}$ and variance $E\sigma_v^2 - (E(\sigma_v))^2 = \rho_v$. Thus, $E\sigma_v^2 = \rho_0 + \rho_v \triangleq \rho$.

The uncertainty in $\rho_0$ will be taken into account by using a higher equivalent variance $\rho$. It is necessary to assure A2 when the covariance matrices are used pragmatically, as "robustness tuning knobs". Design equations could be derived for situations with correlations between $\Delta C$ and $\Delta B$. Assumption A1 does, however, simplify the solution, and seems reasonable.

Model error covariances may be obtained from identification experiments, or from frequency domain data on system variability. See Sternad and Ahlén (1993), Goodwin and Salgado (1989) and Goodwin, Gevers and Ninnes (1991). If a fixed filter is to be designed for a large number of systems, the statistics may be obtained from a representative sample of systems.

Probabilistic error models remain useful also when statistics is hard to obtain. Those who prefer a Bayesian view could then interprete error distributions as subjective probabilities. Others may just use them pragmatically, as robustness "tuning knobs". The covariances are then altered until satisfactory spectral properties of the filter are obtained.

## 4. DESIGN OF ROBUST FILTERS

We proceed from the model (0.1), (0.2) and (0.3). The coefficients of $\Delta C$, $\Delta B$ and $\Delta M$ are random variables, whose possible values parametrize a set of systems. We will minimize the *averaged MSE criterion*

$$\bar{E}(E|z(t)|^2) = \bar{E}E|f(t) - \hat{f}(t|t+m)|^2 \tag{5}$$

where $E$ represents expectation over noise and $\bar{E}$ is an expectation over the model error distribution. We thus seek a single estimator which provides the best MSE

[1] Note that we are talking about an extended design *model.* In practice, it will only be an approximation of a class of possibly infinite dimensional and time–varying true systems.

[2] If unstable poles were exactly known, a finite estimation error could be obtained, by a filter which cancels unstable poles by zeros in the total signal path to the estimation error. Such a strategy is, of course, highly non–robust to mis–modelling of unstable poles. With *uncertain* unstable poles, the design problem becomes unsolvable, in the open–loop context considered here. Therefore, a general solution involving two coupled Diophantine equations will not be of interest here.

performance, on average, when applied on randomly selected systems within the specified class.

This type of criterion has been used in connection to other filtering problems, e.g. by Chung and Bélanger (1976), Speyer and Gustafson (1975) and by Grimble (1984). These works were based on the assumption of small uncertainties and on series expansion of uncertain poles. In our design philosophy, we start from a model structure (0.3), and adjust it to the uncertainty directly. Large uncertainties can then be described in a much better way. See Sternad and Ahlén (1993).

## 4.1 The averaged spectral factorization

An averaged spectral factor $\beta(q^{-1})$ is defined as the stable and monic solution to

$$r\beta\beta_* \triangleq \bar{E}\{CC_*BB_*NN_* + \sigma_v^2 MM_*AA_*DD_*\} \quad (6)$$

with scalar $r$. Define double–sided polynomials

$$\tilde{C}\tilde{C}_* \triangleq \bar{E}(CC_*),\ \tilde{B}\tilde{B}_* \triangleq \bar{E}(BB_*),\quad \tilde{M}\tilde{M}_* \triangleq \bar{E}(MM_*) \quad (7)$$

Then, use of (0.3) gives

$$\begin{aligned}\tilde{C}\tilde{C}_* &= C_0C_{0*}D_1D_{1*} + D_0D_{0*}C_1C_{1*}\bar{E}(\Delta C\Delta C_*)\\ \tilde{B}\tilde{B}_* &= B_0B_{0*}A_1A_{1*} + A_0A_{0*}B_1B_{1*}\bar{E}(\Delta B\Delta B_*)\\ \tilde{M}\tilde{M}_* &= M_0M_{0*}N_1N_{1*} + N_0N_{0*}M_1M_{1*}\bar{E}(\Delta M\Delta M_*)\end{aligned} \quad (8)$$

We can now simplify (0.6).

**Lemma 1.** Let assumptions A1 and A3 hold. Then, (0.6) can be expressed as

$$r\beta\beta_* = \tilde{C}\tilde{C}_*\tilde{B}\tilde{B}_*NN_* + \rho\tilde{M}\tilde{M}_*AA_*DD_* \quad (9)$$

■

**Proof:** The coefficients of a polynomial $\Delta P$ are zero mean stochastic variables. Coefficients of $\Delta P\Delta P_*$ will also be stochastic variables, having expected values given by (0.11) below. The noise standard deviation is independent of $\Delta M$ and the coefficients of $(\Delta B,\ \Delta C)$, are independent, and so are the coefficients of $\Delta B\Delta B_*$, $\Delta C\Delta C_*$. Using independence for complex parameters, the right–hand side of (0.6) becomes

$$\bar{E}(CC_*)\bar{E}(BB_*)NN_* + \bar{E}(\sigma_v)^2\bar{E}(MM_*)AA_*DD_*$$

which, utilizing (0.7) and A3, is (0.9) ■

The averaged factors in (0.8) can be evaluated as follows. For a stochastic error model numerator $\Delta P(q^{-1})$, as in (0.4), let the Hermitian parameter covariance matrix be

$$\mathbf{P}_{\Delta P} = \begin{bmatrix} \bar{E}|\Delta p_0|^2 & \dots & \bar{E}(\Delta p_0\Delta p_{\delta p}^*) \\ \vdots & \ddots & \vdots \\ \bar{E}(\Delta p_{\delta p}\Delta p_0^*) & \dots & \bar{E}|\Delta p_{\delta p}|^2 \end{bmatrix} \quad (10)$$

Denote the sum of the diagonal elements $h_0$, the sum of elements in the $i$'th super–diagonal $h_i$, and the sum of elements in the $i$'th subdiagonal $h_{-i}$. Note that $h_{-i} = h_i^*$. Then it becomes evident, by direct multiplication of $\Delta P(q^{-1})\Delta P_*(q)$, and taking expectations, that

$$\bar{E}(\Delta P\Delta P_*) =$$

$$h_{dp}^* q^{-dp} + \ldots + h_1^* q^{-1} + h_0 + h_1 q + \ldots + h_{dp}q^{dp} \quad . \quad (11)$$

Thus, the averaged factors in (0.8) are readily obtained. (Above, $dp \le \delta p$, with $dp = 0$ if coefficients are uncorrelated.)

In (0.8), $\tilde{C}\tilde{C}_*$ will contain powers up to $q^{\pm n\tilde{c}}$, where $n\tilde{c} = \max\{nc_0 + nd_1, nd_0 + nc_1 + dc\}$, with analogous expressions for $n\tilde{b}, n\tilde{m}$. Since $N = N_0N_1$ etc, the averaged spectral factor in (0.9) has degree

$$n\beta = \max\{n\tilde{c}+n\tilde{b}+nn_0+nn_1, n\tilde{m}+na_0+na_1+nd_0+nd_1\}$$

The factorization (0.9) is solvable with respect to a unique stable $\beta(z^{-1})$ iff its right–hand side is positive on $|z| = 1$. Introduce the assumptions

**A4.** $C_0$, $C_1\bar{E}(\Delta C\Delta C_*)$, $\rho M_0$ and $\rho M_1\bar{E}(\Delta M\Delta M_*)$ have no common zeros on $|z| = 1$

**A5.** $B_0$, $B_1\bar{E}(\Delta B\Delta B_*)$, $\rho M_0$ and $\rho M_1\bar{E}(\Delta M\Delta M_*)$ have no common zeros on $|z| = 1$.

**Lemma 2.** Let $D, A$ and $N$ be stable and A2 hold. Then, a unique stable spectral factor $\beta$, satisfying (0.9), exists, if and only if both of A4 and A5 are true ■

**Proof:** See Sternad and Ahlén (1993).

The conditions A4 and A5 are mild. They will almost always be fulfilled, even if $C_0, B_0$ and $M_0$ have zeros on the unit circle. In fact, the conditions are more relaxed than for the nominal case, due to the presence of averaged factors $\bar{E}(\cdot)$.

## 4.2 The cautious Wiener filter

**Theorem 1.** Assume an extended design model (0.1)–(0.3) to be given, with known covariances of the stochastic polynomial coefficients. Assume A1–A5 to hold. An estimator of $u(t)$ then minimizes (0.5), among all linear time–invariant estimators based on $y(t+m)$, if and only if it has the same coprime factors as

$$\hat{u}(t|t+m) = \frac{Q}{R}y(t+m) \quad ; \quad \frac{Q}{R} = \frac{Q_1N_0N_1A_0A_1}{\beta} \quad (12)$$

Here, $\beta(q^{-1})$ is obtained from (0.9), while $Q_1(q^{-1})$, together with $L_*(q)$, is the unique solution to

$$q^{-m+k}\tilde{C}\tilde{C}_*B_{0*}A_{1*}N_{0*}N_{1*} = r\beta_*Q_1 + qD_0D_1L_* \quad (13)$$

with polynomial degrees

$$\begin{aligned} nQ_1 &\leq \max(n\tilde{c} - k + m \ , \ nd_0 + nd_1 - 1) \\ nL &\leq \max(n\tilde{c} + nb_0 + na_1 + nn_0 + nn_1 \\ &\quad +k - m \ , \ n\beta) - 1 \ . \end{aligned} \tag{14}$$

For an ensemble of systems, the minimal criterion value becomes

$$\bar{E}E|z(t)|^2_{\min} = \frac{1}{2\pi i}\oint \frac{LL_*}{r\beta\beta_*} + \rho\frac{\tilde{C}\tilde{C}_*\tilde{M}\tilde{M}_*AA_*}{r\beta\beta_*} + \\ + \frac{\tilde{C}\tilde{C}_*\tilde{C}\tilde{C}_*\bar{E}(\Delta\mathcal{G}\Delta\mathcal{G}_*)AA_*NN_*}{DD_*r\beta\beta_*}\frac{dz}{z} \tag{15}$$

■

**Proof:** See Sternad and Ahlén (1993).

**Remarks.** The equations for minimizing (0.5) are (0.9), (0.11) and (0.13). The only new type of computation, as compared to a nominal solution, is trivial: summation of covariance matrix elements, diagonalwise.

Note that $N_1$ and $A_1$ affect the filter (0.12) directly. If $1/N_1$ or $1/A_1$ in the error models have resonance peaks, indicating large uncertainty, the filter (0.12) will have low gain at those frequencies. With increasing model uncertainty, the zeros of $\beta$ are moved inward in the unit circle. Resonance peaks of the estimator are lowered and broadened.

Equation (0.13) will have a unique solution, with degrees (0.14). Note that $\beta_*(z)$ (unstable) and $D_0(z^{-1})D_1(z^{-1})$ (stable) have no common factors.

The integrand of (0.15) consists of three terms. Term 1 represents the effect of finite smoothing lag $m$. It can be shown that $L_*(q) \to 0$ when $m \to \infty$. The second term mainly represents the effect of noise. It vanishes for $\rho = 0$. Finally, the third term represents degradation caused by errors $\Delta\mathcal{G} = B_1\Delta B/A_1$ in the transducer model. It vanishes only when $\Delta\mathcal{G} = 0$.

In situations with little noise and sufficiently large smoothing lag $m$, term 3 in (0.15) will dominate the error. This is not surprising; a deconvolution smoother then essentially inverts $\mathcal{G}$. This operation is sensitive to model errors there.

### 4.3 Analytical expressions for performance evaluation

**Theorem 2.** Let a nominal estimator $Q_0/R_0$ be designed based on a nominal model, as in eg. Ahlén and Sternad (1989),(1991). Applying it, instead of (0.12), on an ensemble of systems results in an increase, compared to (0.15), of the mean MSE $\bar{E}E|z(t)|^2$. The increase is given by

$$\bar{E}E|z(t)|^2_0 - \bar{E}E|z(t)|^2_{\min} = \\ = \frac{r}{2\pi i}\oint \left|\frac{\beta}{DAN}\right|^2 \left|\frac{Q_0}{R_0} - \frac{Q}{R}\right|^2 \frac{dz}{z} \tag{16}$$

where $r, \beta$ is defined by (0.6)-(0.9) and $Q/R$ is the optimal robust filter, given by (0.12). ■

**Proof:** See Sternad and Ahlén (1993).

**Theorem 3.** Let a robust estimator $Q/R$ be designed by (0.9)–(0.13). When applying it on a system equal to the nominal model, the increased MSE, compared to the minimum, is

$$E|z(t)|^2 - E|z(t)|^2_0 = \\ \frac{r_0}{2\pi i}\oint \left|\frac{\beta_0}{D_0A_0N_0}\right|^2 \left|\frac{Q}{R} - \frac{Q_0}{R_0}\right|^2 \frac{dz}{z} \tag{17}$$

where $r_0$ and $\beta_0$ are obtained from the spectral factorization in a nominal design. ■

**Proof:** See Sternad and Ahlén (1993).

**Remarks.** The expression in (0.16) can be used for arbitrary linear estimators $Q_0/R_0$, for example minimax–designs. We thus do not have to evaluate the mean performance of alternative designs by Monte–Carlo simulation. The mean innovations model $\beta/DAN$ in (0.16), and the nominal innovations model $\beta_0/D_0A_0N_0$ in (0.17) can be seen as weighting functions. In frequency regions where their magnitude is large, differences between the two estimators will have a large impact on the performance.

A simulation study of the filtering performance is presented in Sternad and Ahlén (1993).

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