# ON EFFICIENT IMPLEMENTATION OF THE 2-D CAPON ALGORITHM

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## ABSTRACT

We present an efficient implementation of the 2-D Amplitude Spectrum Capon (ASC) estimator, denoted the 2-D BASC estimator. The implementation, which will depend only on the (forward) linear prediction matrices and the (forward) prediction error covariance matrices, can be implemented using the 2-D Fast Fourier Transform. To compute the needed prediction matrices, we make use of a recently proposed 2-D lattice algorithm, which computes the linear prediction matrices directly from the multichannel data without first computing the autocorrelation sequence.

## 1. INTRODUCTION

The problem of two-dimensional (2-D) high resolution spectral estimation has been widely studied in the past literature [1], as well as in more recent contributions such as [2, 3, 4]. Applications occur in a wide variety of fields, such as geophysics, radio astronomy, biomedical engineering, sonar, and radar, to mention a few. In many of these applications, it is of key importance to obtain computationally efficient high resolution estimates, as for example it is in synthetic aperture radar (SAR) image formation and target feature extraction. Popular approaches include the 2-D Periodogram, and in the higher resolution cases the 2-D AR and the 2-D Capon spectral estimators. A number of approaches have been suggested for efficient estimation of the 2-D AR spectrum (see, e.g., [1, 5, 6]), whereas only limited efforts have been made to simplify the 2-D Capon estimator [1, 3, 4].

In this paper, we present a computationally efficient implementation of the 2-D Amplitude Spectrum Capon (ASC) spectral estimation algorithm. The ASC estimator differs from that of the "classical" Power Spectrum Capon (PSC) in the way the spectral amplitude is estimated. Recent studies have shown that the ASC spectral estimator will have significantly higher resolution than the PSC spectral estimator [7, 8]. Another important difference is that the ASC estimate will retain the signal's phase information, which the PSC estimate will not. The phase information is often needed, for instance in SAR imaging where one may use the phase to improve the final image or to determine the height information in the image. The proposed implementation is based on the recent 2-D extension of the famous Burg algorithm [9, 4], and is denoted the 2-D BASC. The proposed implementation, which is a 2-D extension of the 1-D BASC algorithm presented in [7], will depend only on the (forward) linear prediction matrices and can be efficiently implemented using the 2-D Fast Fourier Transform (FFT). Note that if one has no interest in the signal's phase information, and the resolution of the 2-D PSC method is adequate, one does better by using the 2-D PSC algorithm presented in [4] as it is significantly faster than the here presented 2-D ASC algorithm.

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## 2. THE MATCHED FILTERBANK APPROACH

In the filterbank approach to spectral estimation, the spectrum is estimated by passing the measured signal through an $(L_1+1,L_2+1)$-tap 2-D narrowband finite impulse response (FIR) filter, $\mathbf{H}_{\omega_1,\omega_2}$, with varying center frequencies $(\omega_1,\omega_2)$ (see, e.g.,[2]). Let the $N_1 \times N_2$ data matrix $\mathbf{Z}$ denote the available (stationary) 2-D data sample of which the spectrum is to be estimated. The filter output can then be written as

$$\mathbf{h}^*_{\omega_1,\omega_2}\mathbf{y}_{t,s} = \alpha_{\omega_1,\omega_2}e^{i(t\omega_1+s\omega_2)} + n_{t,s} \tag{1}$$

for $t = 0,\dots,M_1$, $s = 0,\dots,M_2, \omega_1,\omega_2 \in [0,2\pi)$, where $(\cdot)^*$ and $n_{t,s}$ denote the complex conjugate transpose and some additive colored noise term. Here $M_1 = N_1 - L_1 - 1$, $M_2 = N_2 - L_2 - 1$ and the $(L_1+1)(L_2+1)\times 1$ filter vector $\mathbf{h}_{\omega_1,\omega_2} = \text{vec}(\mathbf{H}_{\omega_1,\omega_2})$, where $\text{vec}(\cdot)$ denotes the operation of stacking the columns of a matrix on top of each other. Similarly, the $(L_1+1)(L_2+1)\times 1$ snapshot vector, $\mathbf{y}_{t,s}$, is defined as $\mathbf{y}_{t,s} = \text{vec}(\mathbf{Y}_{t,s})$, where the $(L_1+1)\times(L_2+1)$ submatrices $\mathbf{Y}_{t,s}$ are defined as

$$\mathbf{Y}_{t,s} = \begin{bmatrix} \mathbf{Z}_{t+L_1,s+L_2} & \dots & \mathbf{Z}_{t,s+L_2} \\ \vdots & \ddots & \vdots \\ \mathbf{Z}_{t+L_1,s} & \dots & \mathbf{Z}_{t,s} \end{bmatrix} \tag{2}$$

for $t = 0,\dots,M_1$, $s = 0,\dots,M_2$. The least-squares estimate, $\hat{\alpha}_{\omega_1,\omega_2}$, of the complex amplitude, $\alpha_{\omega_1,\omega_2}$, in (1) is given by

$$\hat{\alpha}_{\omega_1,\omega_2} = \mathbf{h}^*_{\omega_1,\omega_2}\mathbf{G}_{\omega_1,\omega_2}, \tag{3}$$

where

$$\mathbf{G}_{\omega_1,\omega_2} = \frac{1}{M_1M_2}\sum_{k_1=0}^{M_1}\sum_{k_2=0}^{M_2}\mathbf{y}_{k_1,k_2}e^{-i(k_1\omega_1+k_2\omega_2)}. \tag{4}$$

The problem of designing $\mathbf{h}_{\omega_1,\omega_2}$ as a matched filterbank (MAFI) was studied in [2]. It was found that the 2-D ASC method can be interpreted as being member of the MAFI class. The corresponding filter is given by (see [2] for further details)[1]

$$\mathbf{h}_{\omega_1,\omega_2} = \frac{\hat{\mathbf{R}}^{-1}\mathbf{a}_L(\omega_1,\omega_2)}{\mathbf{a}^*_L(\omega_1,\omega_2)\hat{\mathbf{R}}^{-1}\mathbf{a}_L(\omega_1,\omega_2)}, \tag{5}$$

[1]Note that both the PSC and the ASC spectral estimators are constructed using the same filter. The difference lies only in how the methods estimate the *amplitude spectrum* [10].

where $\hat{\mathbf{R}}$ is an estimate of the sample covariance matrix. Here, $\mathbf{a}_L(\omega_1, \omega_2)$ is the 2-D Fourier vector, defined as

$$\begin{aligned} \mathbf{a}_L(\omega_1, \omega_2) &\triangleq \mathbf{a}_{L_1}(\omega_k) \otimes \mathbf{a}_{L_2}(\omega_k) & (6) \\ \mathbf{a}_{L_k}(\omega_k) &\triangleq \begin{bmatrix} 1 & e^{-i\omega_k} & \ldots & e^{-iL_k\omega_k} \end{bmatrix}^T, & (7) \end{aligned}$$

where $\otimes$ and $(\cdot)^T$ denote the Kronecker product and the transpose, respectively. The *true* sample covariance matrix, $\mathbf{R}$, is defined as

$$\begin{aligned} \mathbf{R} &\triangleq E\left\{\mathbf{y}_{t,s}\mathbf{y}_{t,s}^*\right\} \\ &= \begin{bmatrix} \mathbf{R}_0 & \mathbf{R}_1 & \ldots & \mathbf{R}_{L_1} \\ \mathbf{R}_1^* & \mathbf{R}_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{R}_1 \\ \mathbf{R}_{L_1}^* & \ldots & \mathbf{R}_1^* & \mathbf{R}_0 \end{bmatrix}, \end{aligned} \tag{8}$$

where $E\{\cdot\}$ denotes the expectation, and where the block matrices $\mathbf{R}_k$ are defined as

$$\mathbf{R}_k \triangleq \begin{bmatrix} r_{k,0} & r_{k,1} & \ldots & r_{k,L_2} \\ r_{k,1}^* & r_{k,0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & r_{k,1} \\ r_{k,L_2}^* & \ldots & r_{k,1}^* & r_{k,0} \end{bmatrix}, \tag{9}$$

where $r_{k,l} = E\left\{\mathbf{Z}_{t+k,s+l}\ \mathbf{Z}_{t,s}^*\right\} = r_{-k,-l}^*$. Note that the covariance matrix $\mathbf{R}$ has a (Hermitian) Toeplitz-Block-Toeplitz structure. An estimate of the covariance matrix, $\hat{\mathbf{R}}$, needed in (5), can be obtained in several different ways. We will in our numerical examples use the outer-product forward-backward sample covariance matrix estimate (see also [4, 10]). The 2-D ASC amplitude estimate, at frequencies $(\omega_1, \omega_2)$, is given by (3) evaluated using the filter (5), i.e.,

$$\hat{\alpha}_{\omega_1,\omega_2} = \frac{\mathbf{a}_L^*(\omega_1,\omega_2)\hat{\mathbf{R}}^{-1}\mathbf{G}_{\omega_1,\omega_2}}{\mathbf{a}_L^*(\omega_1,\omega_2)\hat{\mathbf{R}}^{-1}\mathbf{a}_L(\omega_1,\omega_2)}, \tag{10}$$

and the corresponding spectral estimate is given as the magnitude square of (10), i.e.,

$$\hat{\varphi}_{\omega_1,\omega_2} = \left|\hat{\alpha}_{\omega_1,\omega_2}\right|^2. \tag{11}$$

Note that the 2-D ASC will in general yield a different and often preferable spectral estimate than the 2-D PSC spectral estimator, which is obtained by estimating the power of the filter output, i.e.,

$$\hat{\varphi}_{\omega_1,\omega_2}^{PSC} = \mathbf{h}_{\omega_1,\omega_2}^*\hat{\mathbf{R}}\mathbf{h}_{\omega_1,\omega_2} = \frac{1}{\mathbf{a}_L^*(\omega_1,\omega_2)\hat{\mathbf{R}}^{-1}\mathbf{a}_L(\omega_1,\omega_2)}. \tag{12}$$

The problem of interest in this paper is to compute (10) in a computationally efficient manner. We note that the primary computational burden to evaluate (10) is not, as it might first seem, that associated with the inverse of the large dimension matrix $\hat{\mathbf{R}}$. If that was the case, an efficient algorithm for the inversion of a Toeplitz-Block-Toeplitz matrix could have been used [11]. Rather, it is the computation of (10) over all frequencies that is normally more time consuming. In the following section, we propose an efficient way to do this using the 2-D FFT.

## 3. PROPOSED EFFICIENT IMPLEMENTATION

As was suggested in [12, 3], let $\mathbf{C} \triangleq \mathbf{R}^{-1/2}$ denote a square root of the positive definite matrix $\mathbf{R}^{-1}$ defined in (8), and let

$$\begin{aligned} \boldsymbol{\nu}_{\omega_1,\omega_2} &\triangleq \mathbf{C}^*\mathbf{a}_L(\omega_1,\omega_2) & (13) \\ \boldsymbol{\mu}_{\omega_1,\omega_2} &\triangleq \mathbf{C}^*\mathbf{G}_{\omega_1,\omega_2} = \frac{\mathbf{C}^*\mathbf{W}\mathbf{a}_M(\omega_1,\omega_2)}{M_1M_2} & (14) \end{aligned}$$

where

$$\mathbf{W} = \begin{bmatrix} \mathbf{y}_{0,0} & \ldots & \mathbf{y}_{M_1,0} & \ldots & \mathbf{y}_{0,M_2} & \ldots & \mathbf{y}_{M_1,M_2} \end{bmatrix}. \tag{15}$$

Making use of (12), as well as (13), the 2-D PSC spectral estimate can be formulated as (see also [4])

$$\varphi_{\omega_1,\omega_2}^{PSC} = \frac{1}{\boldsymbol{\nu}_{\omega_1,\omega_2}^*\boldsymbol{\nu}_{\omega_1,\omega_2}}. \tag{16}$$

Similarly, the 2-D ASC spectral estimate in (11) can be found as

$$\varphi_{\omega_1,\omega_2}^{ASC} = \left|\frac{\boldsymbol{\nu}_{\omega_1,\omega_2}^*\boldsymbol{\mu}_{\omega_1,\omega_2}}{\boldsymbol{\nu}_{\omega_1,\omega_2}^*\boldsymbol{\nu}_{\omega_1,\omega_2}}\right|^2. \tag{17}$$

Thus, both the 2-D PSC and the 2-D ASC spectral estimators can, given an estimate of $\mathbf{C}$, both be efficiently computed using the 2-D FFT. An efficient computation of (17) was recently proposed in [3]. There, $\mathbf{C}$ was estimated by computing the Cholesky factorization of inverted the outer-product sample covariance estimate $\hat{\mathbf{R}}$. This approach requires the computing of $\hat{\mathbf{R}}$ as well as the Cholesky factor $\hat{\mathbf{C}}$. Here, we instead propose to construct $\mathbf{C}$ from the (forward) linear prediction matrices (see, e.g., [13], Complement C8.3)

$$\begin{aligned} \hat{\mathbf{C}} &= \begin{bmatrix} \mathbf{I} & & & \mathbf{0} \\ \mathbf{A}_{1,L_1}^* & \mathbf{I} & & \\ \vdots & \ddots & \ddots & \\ \mathbf{A}_{L_1,L_1}^* & \ldots & \mathbf{A}_{1,1}^* & \mathbf{I} \end{bmatrix} \times \\ &\times \begin{bmatrix} \mathbf{U}_{L_1}^{-1/2} & & & \mathbf{0} \\ & \mathbf{U}_{L_1-1}^{-1/2} & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{U}_0^{-1/2} \end{bmatrix} \end{aligned} \tag{18}$$

where $\{\mathbf{A}_{k,n}\}$ and $\mathbf{U}_n$ denote the matrix coefficients and prediction error covariance matrices of the (forward) linear prediction model of order $n$ (see [1, 13] for further details). Here, $\mathbf{U}_0 = \mathbf{R}_0$. To compute the needed matrix square roots in (18), we use the Cholesky factorization. As these matrices are significantly smaller than the full covariance matrix $\hat{\mathbf{R}}$, the computational burden of doing so will in comparison be minor. As was shown in [4], one will obtain better spectral estimates if the needed linear prediction matrices are computed using the recent 2-D lattice algorithm [9, 4], as compared to the Whittle-Wiggins-Robinson algorithm (WWRA) (see [13]) based on an estimate of the 2-D covariance matrix. This is well in accordance with similar results in the 1-D case. Note that the 2-D BASC estimator will produce (almost) the same estimate as the 2-D ASC estimator computed from the forward-backward averaged covariance matrix estimate. This forward-backward ASC (FB-ASC) will yield a significantly better spectral estimate, although with a somewhat lower resolution, than the forward-only

ASC estimator [2]. The reason that BASC will produce only *almost* the same estimate as FB-ASC can be explained as follows: the estimate of $\mathbf{C}$, as obtained by using either the WWRA or the 2-D lattice algorithm, will in general *not* yield the same estimate of $\mathbf{C}$ as the Cholesky factorization of the forward-backward covariance matrix estimate. This is due to the fact that $\hat{\mathbf{R}}$ is the outer product covariance matrix estimate, and will thus not have the Toeplitz-Block-Toeplitz structure that is obtained if an estimate of $\mathbf{R}$ is constructed from a $\mathbf{C}$ computed as suggested. Thus, the BASC and the FB-ASC spectral estimates will not be *identical*, although as is shown in the next section the estimates are basically the same.

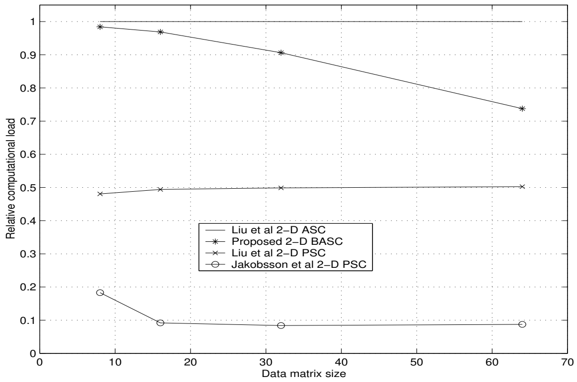


Figure 1: Relative computational complexity vs data matrix size.

## 4. NUMERICAL EXAMPLES

We first study the computational complexity of the different implementations. The data matrix has been generated as the sum of four 2-D complex sinusoids (cisoids) corrupted by additive complex Gaussian white noise. The computational complexity is evaluated as the data matrix size, $N_1 = N_2$, varies. Figure 1 illustrates the relative computational complexity of the proposed 2-D BASC spectral estimator for varying data matrix dimensions, as compared to the 2-D PSC and 2-D ASC methods proposed in [3] as well as the 2-D PSC estimator in [4]. In the figure, the computational load of the different estimators, as measured by MATLAB, has been normalized with the the load of the 2-D ASC method. The simulation shows that the proposed 2-D BASC estimator will be clearly faster than the 2-D ASC estimator in [3], especially for larger matrices. As the difference between the two estimators basically lies in how the $(L_1 + 1) \times (L_2 + 1)$ Cholesky matrix, $\mathbf{C}$, is computed, this comes as no surprise. From the figure, it can also be seen that the 2-D PSC implementation in [4] will be about five times faster than the 2-D PSC implementation in [3]. The reader is reminded that the 2-D BASC algorithm should only be used in cases where one wishes to retain the signal's phase information, or when the resolution of the 2-D PSC algorithm is not sufficient. In the example, the filterlengths were $L_1 = L_2 = N_1/4$, and the spectrum is zeropadded to length $N_{\omega_1} = N_{\omega_2} = 4N_1$ (which means that the spectrum is evaluated using a $4N_1 \times 4N_1$-point 2-D FFT). We proceed by illustrating the spectral resolutions achieved by the different methods. First, we use a $32 \times 32$ data matrix that consists of a sum of four cisoids corrupted by additive complex Gaussian white noise. The cisoids all have unit amplitude, a phase offset of $\pi/4$, and are located at $f = (-0.25, 0.25), (0, 0), (0.03, 0), (0.3, -0.1)$. Figures 2(a)–(c) illustrate the resulting resolutions obtained by using the 2-D PSC and 2-D ASC estimates of [3], as well as the 2-D PSC estimate presented in [4]. The proposed 2-D BASC estimate is shown in Figure 2(d). As seen from the figure, the ASC estimators will have better resolution than the PSC estimators. Also note that the ASC estimates will have a somewhat higher noise floor than PSC estimates. In the example, the filter lengths were $L_1 = L_2 = N_1/4$, and the spectrum is zeropadded to length $N_{\omega_1} = N_{\omega_2} = 4N_1$. Further numerical examples can be found in [10].

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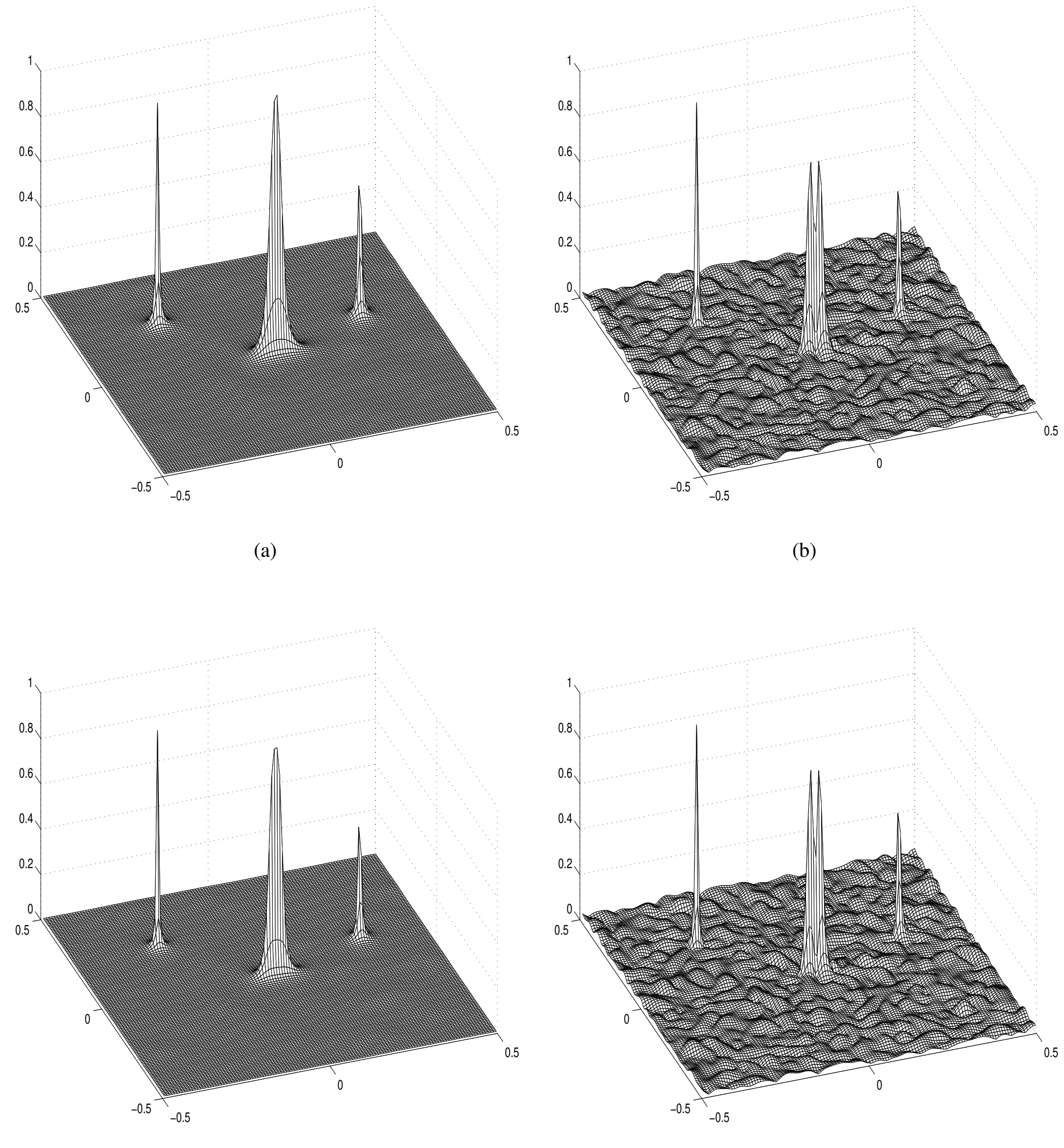


Figure 2: Illustration of the resolution and accuracy for the different spectral estimators. The estimates are plotted for fractions of the sampling frequency. (a) The 2-D PSC estimate of [3]. (b) The 2-D ASC estimate of [3]. (c) The 2-D PSC estimate presented in [4]. (d) The proposed 2-D BASC estimate.